Surface Normals at Singular Points

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Abstract

For a parametric surface $S(u,v) = (X(u,v), Y(u,v), Z(u,v))$ the unit surface normal is conventionally given by the formula $S_u \times S_v / \|S_u \times S_v\|$. This formula may break down at singular points where $S_u \times S_v = 0$. In this paper we show how to decide whether the surface normal exists at a singular point and if so how to calculate it.

1 Introduction

Much of computer graphics is concerned with the definition and rendering of parametric surfaces. We will consider the general case $S : A \to \mathbb{R}^3$

$$S(u,v) = (X(u,v), Y(u,v), Z(u,v)),$$

where A$\subset \mathbb{R}^2$ is a rectangular region $[u_0, u_1] \times [v_0, v_1]$ and X, Y, Z are analytic on A .

In computer graphics, a very basic calculation during the rendering process is the surface normal $N = S_u \times S_v / \|S_u \times S_v\|$ where $S_u = \frac{\partial S}{\partial u}$ and $S_v = \frac{\partial S}{\partial v}$ are the partial derivatives with respect to u and v. During rendering a surface is tessellated into a triangular mesh and the normal is evaluated at vertices of the mesh. For points in the interior of each mesh triangle, an approximation to the true normal is made by interpolating the normals at the triangle's vertices. These approximate normals are used for shading the surface. Surface normals are also used for other purposes, such as constructing offset surfaces.

Points where $S_u \times S_v = 0$ are called *singular*, and a direct evaluation of N at these points results in a divide by zero. Singular points commonly occur, for example, in spherical spline surfaces at the poles. Leaving the normal undefined at such points is not acceptable, because this would lead to undefined pixels during rendering. Setting the normal to zero is equally unacceptable because it will cause the singular points to be shaded as if there were no directional light sources.

Perhaps the most obvious workaround to this problem is to evaluate $S_u \times S_v$ at a nearby non-singular point. This is usually works from a practical standpoint, and is better than the alternative of doing nothing at all. But it leaves open the conceptual question of whether a "real" normal actually exists and what to do when it doesn't.

The existing computer graphics literature does not adequately treat this question. Even when it is raised, as in [1], the suggested workarounds are on a case by case basis. In the differential geometry literature the problem is defined away by defining parametric surfaces as having no singularities. In Section 2 we analyze the question mathematically, and in Section 3 we outline an algorithm. Section 4 contains a number of examples which the reader may want to examine before reading Sections 2 and 3.

2 Mathematical Background

In this section we consider an analytic function $F : A \to \mathbb{R}^3$

$$F(u,v) = (f(u,v), g(u,v), h(u,v)).$$

Let $Z \subset A$ be the set of zeroes of F. We assume that the zeroes in Z are isolated.

Let $\mathbb{S}^2$ be the unit sphere $\{(x,y,z)|x^2+y^2+z^2=1\}$, and define the Gauss map $G : \mathbb{R}^3 \backslash 0 \to \mathbb{S}^2$ as

$$G(x) = x / \|x\|.$$

The central question of this paper is whether the map $GF : A \backslash Z \to \mathbb{S}^2$ can be continuously extended to a map whose domain is all of A. Our motivating case is $F = S_u \times S_v$, where Z is the set of singular points of S, and the normal is $G(S_u \times S_v)$.

The Taylor series expansion for F at the point (u_0, v_0) is:

$$F(u_0+\Delta u, v_0+\Delta v) = F+\Delta u F_u+\Delta v F_v+\frac{(\Delta u)^2}{2}F_{uu}+\Delta u \Delta v F_{uv}+\frac{(\Delta v)^2}{2}F_{vv}+\ldots,$$

where on the right hand side F and its partial derivatives are evaluated at (u_0, v_0).

This is an expansion in all directions around the point (u_0, v_0). Let $\mathbb{S}^1$ be the unit circle $\{(s,t)|s^2+t^2=1\}$, which we can think of as a set of directions. We wish to compute the above Taylor series for a straight line radiating from (u_0, v_0) in a direction $(s,t) \in \mathbb{S}^1$. Letting $\Delta u = rs$ and $\Delta v = rt$,

$$F(u_0+rs, v_0+rt) = F + rsF_u + rtF_v + \frac{r^2 s^2}{2}F_{uu} + r^2 st F_{uv} + \frac{r^2 t^2}{2}F_{vv} + \ldots,$$

and collecting terms according to powers of r,

$$F(u_0+rs, v_0+rt) = D_0(s,t) + rD_1(s,t) + \ldots = \sum_i r^i D_i(s,t),$$

where the $D_i(s,t)$ are vectors of polynomials in s and t. Let $D_n(s,t)$ be the first term in the series $D_i(s,t)$ which is not identically zero. This term determines the *directional limit* of GF as we approach (u_0, v_0) for directions (s, t) for which $D_n(s, t)$ does not vanish:

$$\begin{aligned}\lim_{r\to 0} GF(u_0 + rs, v_0 + rt) &= \lim_{r\to 0} G(r^n D_n(s, t)) \\ &= G(D_n(s, t)) \\ &= D_n(s, t) / \left\| D_n(s, t) \right\|\end{aligned}$$

For values of (s, t) at which $D_n(s, t) = 0$, the first non-zero higher order term $D_{n+m}(s, t)$ will determine the directional limit of GF, unless $F(u_0+rs, v_0+rt) = 0$ (i.e. there is no such higher order term). If the directional limit is identical from all directions for which $F(u_0+rs, v_0+rt) \neq 0$, then we can define the *limit normal* of F at (u_0, v_0) as the common directional limit. The limit normal will in turn exist if $D_n(s, t) = D(s, t)k$, where the k is a constant vector and $D(s, t)$ is a polynomial that does not change sign as (s, t) winds around $\mathbb{S}^1$. Otherwise, if the components of $D_n(s, t)$ are independent polynomials, $D_n(s, t)$ will not point in a constant direction as (s, t) winds around $\mathbb{S}^1$. Note that only a subset of $\mathbb{S}^1$ need be considered for points on the boundary of A.

The above doesn't quite cut it. We have to define dominant normal. It may be good to try to define GF as a mapping from $\mathbb{S}^1 \to \mathbb{S}^2$. This is a mapping of directions to directional limits.

For the motivating example $F = S_u \times S_v$,

$$\begin{aligned}S_u(u_0 + rs, v_0 + rt) &= S_u + rsS_{uu} + rtS_{uv} + \ldots \\ S_v(u_0 + rs, v_0 + rt) &= S_v + rtS_{uv} + rsS_{vv} + \ldots\end{aligned}$$

and

$$\begin{aligned}S_u \times S_v(u_0 + rs, v_0 + rt) &= D_0(s, t) + rD_1(s, t) + \ldots \\ &= \sum_i r^i D_i(s, t).\end{aligned}$$

Note that $D_0(s, t) = S_u \times S_v$, so at non-singular points (where by definition $S_u \times S_v$ is non-zero and independent of (s, t)) the limit normal is equal to the normal given by the conventional formula.

3 The Limit Algorithm

Based on the analysis in the previous section we sketch an algorithm for the determination of the existence of normals. The algorithm assumes symbolic computation capabilities.

Algorithm 1 (Limit Algorithm). Given S in symbolic form, and a point (u_0, v_0):

1. Symbolically calculate $S_u \times S_v(u, v)$.

2. Symbolically calculate $f(r, s, t) = S_u \times S_v(u_0 + rs, v_0 + rt)$.

3. Calculate the Taylor series for $f(r, s, t)$ in powers of r expanded around $r = 0$.

4. Find the first coefficient $D_n(s, t)$ of this Taylor series that is not identically zero.

5. If the x, y, and z components of $D_n(s, t)$ are not scalar multiples of a common polynomial $D(s, t)$, the limit normal does not exist. Otherwise, compute $k = D_n(s, t)/D(s, t)$. Then the candidate limit normal is $k/\|k\|$.

6. For values $(s, t) \in \mathbb{S}^1$ at which $D_n(s, t) = 0$, calculate the directional limit normals. If these are all the same as the candidate limit normal from the previous step, then the limit normal has been computed. Otherwise, the limit normal does not exist. $\square$

4 Examples

Example 4.1. Let S be the spherical surface $(\cos u \sin v, \sin u \sin v, \cos v)$, $u \in [0, 2\pi], v \in [0, \pi]$ Then $S_u \times S_v = \sin v(-\cos u \sin v, -\sin u \sin v, -\cos v)$, and S is singular at $(u_0, v_0) = (0, 0)$, i.e. the pole $(0, 0, 1)$. Intuition tells us that the normal at that point clearly exists and is equal to $\pm(0, 0, 1)$. Examining the formulae for $S_u \times S_v$ and S_u, we see that the "culprit" behind the singularity is the $\sin v$ term. Indeed,

$$S_u \times S_v/\|S_u \times S_v\| = (-\cos u \sin v, -\sin u \sin v, -\cos v),$$

because the $\sin v$ term cancels out. So even though $S_u \times S_v$ vanishes at $(0, 0)$, $S_u \times S_v/\|S_u \times S_v\|$ does not.

Using the Limit Algorithm, we find that

$$D_1(s, t) = t(0, 0, -1).$$

$D_1(s, t)$ is the first non-zero term in the series D_i, and is collinear with $(0, 0, -1)$ for all values $t \neq 0$. If $t = 0$ then $S_u \times S_v$ is zero, so we obtain the limit normal at the point $S(0, 0)$ as $D_1(s, t)/\|D_1(s, t)\| = (0, 0, -1)$. $\square$

Example 4.2. Let S be the cone (standing on its apex at (0,0,0))

$$(v \cos u, v \sin u, v), u \in [0, 2\pi], v \in [0, 1].$$

Then $S_u \times S_v = (v \cos u, v \sin u, -v)$ and S is singular at $(u_0, v_0) = (x, 0)$, i.e. at its apex. Unlike Example 4.1, intuition is a less certain guide to the existence of a normal at that point. The formula for $S_u \times S_v$ tells us that the common factor v lies behind the singularity, and cancels out when we calculate the normal $S_u \times S_v/\|S_u \times S_v\| = (\cos u, \sin u, -1)/\sqrt{2}$, which is defined everywhere.

Using the Limit Algorithm, at the points $(x, 0)$ we calculate

$$D_1(s, t) = t(\cos x, \sin x, -1),$$

so for a given value of x there is a limit normal. But since all values of $S(x,0)$ coincide at $(0,0,0)$ it remains that the apex of the cone does not have a unique normal. If the cone were to be discretized into a triangular mesh, the vertex corresponding to the apex would have a different normal for every face. □

Example 4.3. Let S be the square $(u^3, v^3, 0), u \in [-1,1], v \in [-1,1]$. Then $S_u \times S_v = (0,0,9u^2v^2)$ and S is singular at the point $(u_0, v_0) = (0,0)$. But intuitively the normal exists for the entire square and is $(0,0,1)$. The Limit Algorithm easily yields this fact. □

Example 4.4. Let S be the patch $(u^2, v^2, uv), u \in [0,1], v \in [0,1]$. Then $S_u \times S_v = (-2v^2, -2u^2, 4uv)$. For the expansion at the singular point $(u_0, v_0) = (0,0)$ we read off

$$D_2(s,t) = (-2t^2, -2s^2, 4st).$$

As (s,t) varies, the values of $D_2(s,t)$ are not collinear, so neither the limit normal nor dominant normal exist at the point $S(0,0)$. □

Example 4.5. Let S be the patch $(u^2, v^3, uv), u \in [0,1], v \in [0,1]$. Then $S_u \times S_v = (-3v^3, -2u^2, 6uv^2)$. Expanding at the singular point $(u_0, v_0) = (0,0)$:

$$S_u \times S_v = r^2(0, -2s^2, 0) + r^3(-3t^3, 0, 6st^2).$$

The normal at (u_0, v_0) does not exist, but the dominant normal is equal to $(0,-1,0)$. Note that the directional normal for $(s,t) = (0,1)$ is $(-1,0,0)$. □

Example 4.6. Let F be the vector field $(u+v, \sin(u+v), u^3+v^3), u \in [0,1], v \in [0,1]$. At the singular point $(u_0, v_0) = (0,0)$, $D_1(s,t) = (s+t, s+t, 0)$. In directions $s+t = 0$, F also vanishes, so $(1,1,0)/\sqrt{2}$ is the limit normal. We note that, unlike Examples 4.1, 4.2, and 4.3, the normal at $(u_0, v_0) = (0,0)$ cannot be determined by simplifying $S_u \times S_v / \|S_u \times S_v\|$. □

Example 4.7. Let S be the patch $(u^2, v^2, 0), u \in [0,1], v \in [0,1]$. At the singular point $(u_0, v_0) = (0,0)$, $D_1(s,t) = (0,0,4st)$ and the normal is evidently $(0,0,1)$. If the domain for this patch was $[-1,1] \times [-1,1]$ instead of $[0,1] \times [0,1]$, the normal would not exist because $(0,0,4st)$ changes direction as (s,t) winds around $\mathbb{S}^1$. Note that such a patch is degenerate in the sense that it is not a 1-to-1 mapping. □

References

[1] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design.* Academic Press, 1988.